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On some questions related to integer lattice points on the plane.

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1.INTRODUCTION

Several questions related to integer lattice points on the plane occupy central position in Analytic Number Theory and Geometry of Numbers. They include the problem of estimating the error term in counting the number of integer lattice points in regions; the case of the region bounded by a circle is the famous 'circle problem' originating in the work of Gauss. But apart from famous problems like this, there are other interesting open questions related to integer lattice points on the plane - some of them not very well known - deserving our attention.

The monograph [9] 'Lattice Points' of Erdös, Gruber and Hammer is an excellent survey of results and open questions, of a wide variety, related to lattice points and they include many questions related to integer lattice points on the plane. In the present expository article, we pick up two themes among the ones that appear in the above mentioned monograph and discuss about the progress made towards obtaining answers to some of the questions therein.

2.VISIBILITY OF LATTICE POINTS

Our first theme is 'visibility' of an integer lattice point from another.

Definitions and trivial observations: Let $P=(a,b)$ and $Q=(m,n)$ be integer lattice points on the xy-plane with rectangular cartesian co-ordinates. P and Q are said to be visible from each other if the line segment which joins them contains no other lattice point between the end points P and Q.

Clearly, (a,b) and (m,n) are visible from each other if and only if $(a-m,b-n)$ is visible from the origin $(0,0)$.

Now, if $\gcd(a, b) = d > 1$, then writing $a = da'$, $b = db'$ one observes that the point (a', b') is on the line segment which joins $(0, 0)$ to (a, b) and hence (a, b) is not visible from the origin. Thus (a, b) is visible from the origin implies that $(a, b) = 1$. Conversely, it is also easy to see that $\gcd(a, b) = 1$ implies that there is no other integer lattice point in the segment joining $(0, 0)$ to (a, b).

Thus, (a, b) is visible from $(0, 0)$ if and only if $\gcd(a, b) = 1$ and hence, by our earlier observation, it follows that (a, b) and (m, n) are mutually visible from each other if and only if $\gcd(a - m, b - n) = 1$.

Further observations and remarks: One observes that there are infinitely many integer lattice points visible from the origin. Next, one tries to find out how the visible lattice points are distributed on the xy-plane.

Let $N'(r)$ be the number of integer lattice points in the square $\{(x, y) : |x| \leq r, |y| \leq r\}$ which are visible from the origin.

It is clear that

$$N'(r) = 8 \sum_{1 \leq n \leq r} \phi(n)$$

where for a positive integer n, $\phi(n)$ is the Euler's totient defined to be the number of positive integers not exceeding n which are prime to n. Therefore, the problem of estimating the quantity $N'(r)$ is the same as that of estimating the average $\sum_{1 \leq n \leq r} \phi(n)$.

Regarding the error term $R(x)$ in the equation

$$\sum_{1 \leq n \leq x} \phi(n) = \frac{3}{\pi^2} x^2 + R(x)$$

the best known O- and Ω - results are due to Walfisz [19] and Montgomery [14] respectively:

$$R(x) = O(x (\log x)^{2/3} (\log \log x)^{4/3})$$

$$R(x) = \Omega_{\pm}(x \sqrt{\log \log x}).$$

Regarding open questions in this direction, the conjectures of Montgomery [14] that $R(x) = O(x \log \log x)$ and $R(x) = \Omega_{\pm}(x \log \log x)$ are yet to be settled.

If $N(r)$ denotes the total number of integer lattice points in the square $\{(x,y) : |x| \leq r, |y| \leq r\}$, then

$$\frac{N'(r)}{N(r)} = \frac{\frac{24r^2}{\pi^2} + O(r\log r)}{4r^2 + O(r)} \longrightarrow \frac{6}{\pi^2} \quad \text{as} \quad r \to \infty.$$

This is expressed by saying that the density of integer lattice points visible from the origin is $\frac{6}{\pi^2}$. In other words, a lattice point chosen at random has the probability $\frac{6}{\pi^2}$ of being visible from the origin.

Some questions: In this section, after mentioning some more interesting results and questions regarding visibility, we shall elaborate on a particular question.

Herzog and Stewart [10] have given criteria for a pattern of visible and nonvisible points being realizable. Erdös, Gruber and Hammer [9] have raised the question of detailed study of the visibility graph. For details of these results and questions one may look into the monograph of Erdös, Gruber and Hammer [9].

Now we come to a particular question we are interested in. Let $\Delta_n = \{(x,y) : x, y$ are integers $, 1 \leq x, y \leq n\}$ be the $n \times n$ square array of integer lattice points in the plane. Let A, B be subsets of Δ_n. A is said to be visible from B if each point of A is visible from some point of B.

$$\text{Let } \ f(n) \stackrel{\text{def}}{=} \min \{|S| : S \subset \Delta_n, \Delta_n \text{ is visible from } S\}.$$

(In the above definition and what follows, for a finite set X, $|X|$ will denote the number of elements in the set.)

In 1974, Abbott [1] proved that

$$\forall n > n_0, \quad \frac{\log n}{2\log\log n} < f(n) < 4\log n. \qquad \cdots\cdots (*)$$

The idea behind the proof of the first inequality is that given any $\frac{\log n}{2\log\log n}$ points of Δ_n, by Chinese Remainder Theorem it follows that there is a point in Δ_n which is not visible from any of the given points. The other inequality follows by using Greedy algorithm.

In the monograph [9] of Erdös, Gruber and Hammer, regarding the above result it has been remarked: "Abbott's proof is an existance proof and gives no indication how to constract small subsets from which any point of the set is visible. It would be of interest to construct such subsets of cardinality $O(\log n)$".

However, the monograph did not mention another result in Abbott's paper [1]. In fact, Abbott had given explicit construction of a set S_n from which Δ_n is visible, but S_n does not meet the requirement of having its cardinality $O(\log n)$. More precisely, Abbott's construction S_n is of the same order as $g(n)$ where $g(n)$ is the Jacobsthal's function defined to be the least integer with the property that among any $g(n)$ consecutive integers $a+1, \cdots, a+g(n)$, there is at least one which is relatively prime to n. Erdös [8] was first to establish that $g(n) = O((\log n)^\alpha)$ for some finite α. Since then several mathematicians (see, for example, [12] [11] [17] [18]) have taken up the problem of improving the estimate of Erdös. Even though it is expected that $g(n) = O((\log n)^{1+\epsilon})$, for any $\epsilon > 0$, it seems that to prove $g(n) = O((\log n)^\alpha)$ with some $\alpha < 2$ would be very difficult. But, in connection with our problem, even if the expected order of $g(n)$ is established, Abbot's explicit construction will fall short of our target. In [2], Adhikari and Balasubramanian could give explicit construction of a set $X_n \subset \Delta_n$ from which Δ_n is visible, where X_n satisfies

$$|X_n| = O\left(\frac{\log n . \log\log\log n}{\log\log n}\right).$$

One observes that the order of $|X_n|$ not only satisfies $(*)$, it improves on $(*)$ by improving the upper bound of $f(n)$. The basic idea in [2] is that by translating an integer by a 'small' amount, one hits upon an integer n, sum of the reciprocals of whose prime divisors is strictly less than one and thereby implying that $g(n)$ is 'small'.

3.A PROBLEM OF STEINHAUS

Our second theme is around a question of Steinhaus. A point set is said to satisfy the Steinhaus property if no matter how it is placed on the plane the number of integer lattice points covered by it is always one.

In what follows, a set satisfying Steinhaus property will be called a Steinhaus set. The main open question here is whether such a set exists or not.

Apart from other related results, Niven and Zuckerman [15] proved that if a compact subset of the plane is freely rotated and translated on the plane, then the number of integer lattice points covered by it at different positions can not be constant. In particular, a compact subset of the plane can not be a Steinhaus set. Surprisingly, the paper of Niven and Zuckerman [15] does not refer to an earlier paper of Sierpinski in which it had been shown that neither a compact nor a bounded open set can be a Steinhaus set.

Later, using harmonic analysis, Beck [5] proved that no bounded measurable set can be a Steinhaus set. Here again, Beck as well as some later writers were aparently unaware of a paper of Croft [7] where a different proof of the above result had already been given.

In 1994, Mihai Ciucu [6] proved that any set having Steinhaus property has empty interior. As a corollary, Ciucu also deduced that a Steinhaus set can not be closed, thus removing the boundedness condition in the results of Sierpinski.

In [3] the ideas of Chicu were pushed to yield finer results. More precisely, it was proved that if a set satisfying Steinhaus property contains a circle of positive radius, then it contains the disc enclosed by the circle. In the light of Mihai Chicu's result, it then follows that a Steinhaus set can not contain a circle of positive radius in it. It was further conjectured [3] that a Steinhaus set can not contain any homeomorphic image of the unit circle. Recently this conjecture has been proved [4].

In another recent paper [13], pushing Beck's methods to its limits, Kolountzakis has proved that if a plane set is measurable and there exists a direction such that the set is 'very small' outside large strips parallel to that direction, then the set can not have Steinhaus property. The main question whether there is a Steinhaus set or not is yet to be settled.

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